Electricity
and
System theory
for Electrochemists

Erase

January 2, 2003

Contents

Part I

Transfer function

Chapter 1

Graphs of transfer function

1.1 Introduction

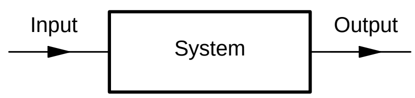

Figure 1.1: Sketch of a scalar system.

- The transfer function, H, of a invariant scalar linear system is given by:

  $$H(s) = \frac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]}$$

  $\mathcal{L}$ denotes the Laplace transform, s is the Laplace variable with $s = \sigma + \mathrm{i}\,\omega$.

- A transfer function is a complex fonction $H(s)$ of two real variables σ and ω. It is not possible to plot graph of $H(s)$ in a plane.

- For $s = \mathrm{i}\,\omega$, *i.e.* $\sigma = 0$, corresponding to frequencial analysis, a transfer function is a complex fonction $H = H(\mathrm{i}\,\omega)$ (or $H(\omega)$) of a real variable ω. It is possible to plot graph of $H = H(\omega)$ in a plane and different types of graph can be used.

- Transfer function order is the degree in s (or $\mathrm{i}\,\omega$) of the transfer function denominator.

- Poles of transfer function are the roots of the denominator of the transfer function $H(s)$.

- Zeros of transfer function are the roots of the numerator of the transfer function $H(s)$.

1.2 Nyquist diagram

1.2.1 Nyquist diagram used by electricians

Orthonormal parametric plot

$$x = \text{Re}\, H = f(\omega)\ ,\ y = \text{Im}\, H = g(\omega) \tag{1.1}$$

1.2.2 Nyquist diagram used by electrochemists

Orthonormal parametric plot

$$x = \text{Re}\, H = f(\omega)\ ,\ y = -\text{Im}\, H = g(\omega) \tag{1.2}$$

1.3 Bode diagram

1.3.1 Bode diagram used by electricians

- Modulus diagram: $20\,\log|H|$ *vs.* $\log\omega$. $|H|$ is the modulus (or magnitude or amplitude) of H with $|H| = \sqrt{(\text{Re}\, H)^2 + (\text{Im}\, H)^2}$.
- Phase diagram: ϕ_H *vs.* $\log\omega$. ϕ_H is the phase of H with $\phi_\text{H} = \arctan\dfrac{\text{Im}\, H}{\text{Re}\, H}$

1.3.2 Bode diagram used by electrochemists

$$\log|H|\ vs\ \log\omega,\ \phi_\text{H}\ vs.\ \log\omega \tag{1.3}$$

1.4 Black diagram

Parametric plot

$$x = \phi_\text{H} = f(\omega)\ ,\ y = 20\,\log|H| = g(\omega) \tag{1.4}$$

Not used, to the best of our knowledge, by electrochemists.

1.5 Miscellaneous

- $\text{Re}\, H$ *vs.* $\log\omega$, $\text{Im}\, H$ *vs.* $\log\omega$
- $\log \text{Im}\, H$ *vs.* $\log \text{Re}\, H$ [?], $\log|\text{Im}\, H|$ *vs.* $\log|\text{Re}\, H|$

Chapter 2

First-order and generalized first-order transfer functions

2.1 First-order transfer function [?]

2.1.1 First-order transfer function

$$H(s) = \frac{K}{1+\tau s},\ H(\omega) = \frac{K}{1+\mathrm{i}\,\omega\,\tau}$$

K: static gain, τ: time constant.

2.1.2 Dimensionless first-order transfer function

$$H^*(S) = \frac{H(s)}{K} = \frac{1}{1+S},\ S = \tau\,s = \Sigma + \mathrm{i}\,u,\ \Sigma = \tau\,\sigma,\ u = \tau\,\omega$$

One real pole: $S_{\mathrm{p}} = -1$ (Fig. 2.1).

$$H^*(u) = \frac{H(\omega)}{K} = \frac{1}{1+\mathrm{i}\,u},\ u = \tau\,\omega \tag{2.1}$$

u: reduced (or dimensionless or nondimensional) angular (or radial) frequency

$$\mathrm{Re}\ H^*(u) = \frac{1}{1+u^2},\ \mathrm{Im}\ H^*(u) = -\frac{u}{1+u^2},\ \lim_{u\to 0} \mathrm{Re}\ H^*(u) = 1$$

Characteristic frequency: $u_{\mathrm{c}} = 1$ (Fig. 2.1).

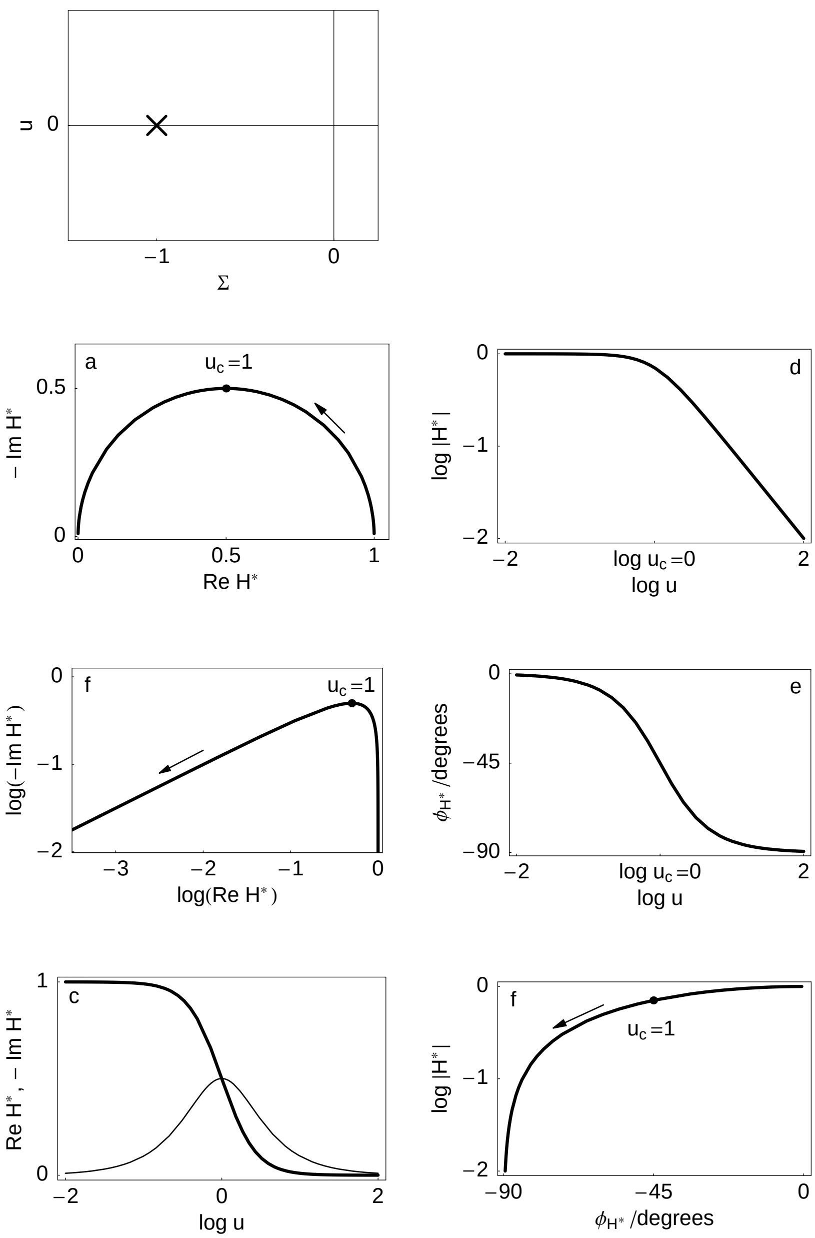

Figure 2.1: Cpz, Nyquist (a), log Nyquist (b) Re H^* *vs.* $\log u$ (c, thick line), $-\mathrm{Im}\ H^*$ *vs.* $\log u$ (c, thin line), Bode (modulus (c) and phase (d)) and Black diagrams of the first order transfer function. Arrow always indicates increasing angular frequencies.

2.2 Generalized first-order transfer functions

2.2.1 High-pass first-order transfer function

$$H(s) = \frac{K\,\tau_{\mathrm{N}}\,s}{1+\tau_{\mathrm{D}}\,s},\ H(\omega) = \frac{K\,\tau_{\mathrm{N}}\,\mathrm{i}\,\omega}{1+\tau_{\mathrm{D}}\,\mathrm{i}\,\omega}$$

2.2.2 Dimensionless high-pass first-order transfer function

$$H^*(S) = \frac{H(s)}{K\,r_\tau} = \frac{S}{1+S},\ r_\tau = \frac{\tau_{\mathrm{N}}}{\tau_{\mathrm{D}}},\ S = \tau_{\mathrm{D}}\,s = \Sigma + \mathrm{i}\,u,\ \Sigma = \tau_{\mathrm{D}}\,\sigma,\ u = \tau_{\mathrm{D}}\,w$$

One real pole: $S_{\mathrm{p}} = -1$, one zero at the origin: $S_{\mathrm{z}} = 0$ (Fig. **??**).

$$H^*(u) = \frac{H(\omega)}{r_\tau} = \frac{\mathrm{i}\,u}{1+\mathrm{i}\,u},\ u = \tau_{\mathrm{D}}\,w$$

$$\mathrm{Re}\ H^*(u) = \frac{u^2}{1+u^2},\ \mathrm{Im}\ H^*(u) = \frac{u}{1+u^2}$$

$$\lim_{u\to\infty} \mathrm{Re}\ H^*(u) = 1$$

Characteristic frequency: $u_{\mathrm{c}} = 1$ (Fig. **??**).

2.2.3 Generalized first-order transfer function

$$H(s) = \frac{K\,(1+\tau_{\mathrm{N}}\,s)}{1+\tau_{\mathrm{D}}\,s},\ H(\omega) = \frac{K\,(1+\tau_{\mathrm{N}}\,\mathrm{i}\,\omega)}{1+\tau_{\mathrm{D}}\,\mathrm{i}\,\omega}$$

2.2.4 Dimensionless generalized first-order transfer function

$$H^*(S) = \frac{H(S)}{K} = \frac{1+r_\tau\,S}{1+S},\ r_\tau = \frac{\tau_{\mathrm{N}}}{\tau_{\mathrm{D}}},\ S = \tau_{\mathrm{D}}\,s = \Sigma + \mathrm{i}\,u,\ \Sigma = \tau_{\mathrm{D}}\,\sigma,\ u = \tau_{\mathrm{D}}\,w$$

One real pole: $S_{\mathrm{p}} = -1 = -u_{\mathrm{c1}}$, one real zero: $S_{\mathrm{z}} = -1/r_\tau = -u_{\mathrm{c2}}$.

$$H^*(u) = \frac{H(u)}{K} = \frac{1+\mathrm{i}\,r_\tau\,u}{1+\mathrm{i}\,u}$$

$$\mathrm{Re}\ H^*(u) = \frac{1+r_\tau\,u^2}{1+u^2},\ \mathrm{Im}\ H^*(u) = \frac{(-1+r_\tau)\ u}{1+u^2}$$

$$\lim_{u\to 0} \mathrm{Re}\ H^*(u) = 1,\ \lim_{u\to\infty} \mathrm{Re}\ H^*(u) = r_\tau$$

Characteristic frequency: $u_{\mathrm{c1}} = 1,\ u_{\mathrm{c2}} = 1/r_\tau\ (\phi_{u_{\mathrm{c1}}} = \phi_{u_{\mathrm{c2}}})$.

$r_\tau < 1 \Rightarrow$ Capacitive behaviour (Fig. **??**).
$r_\tau > 1 \Rightarrow$ Inductive behaviour (Fig. **??**).

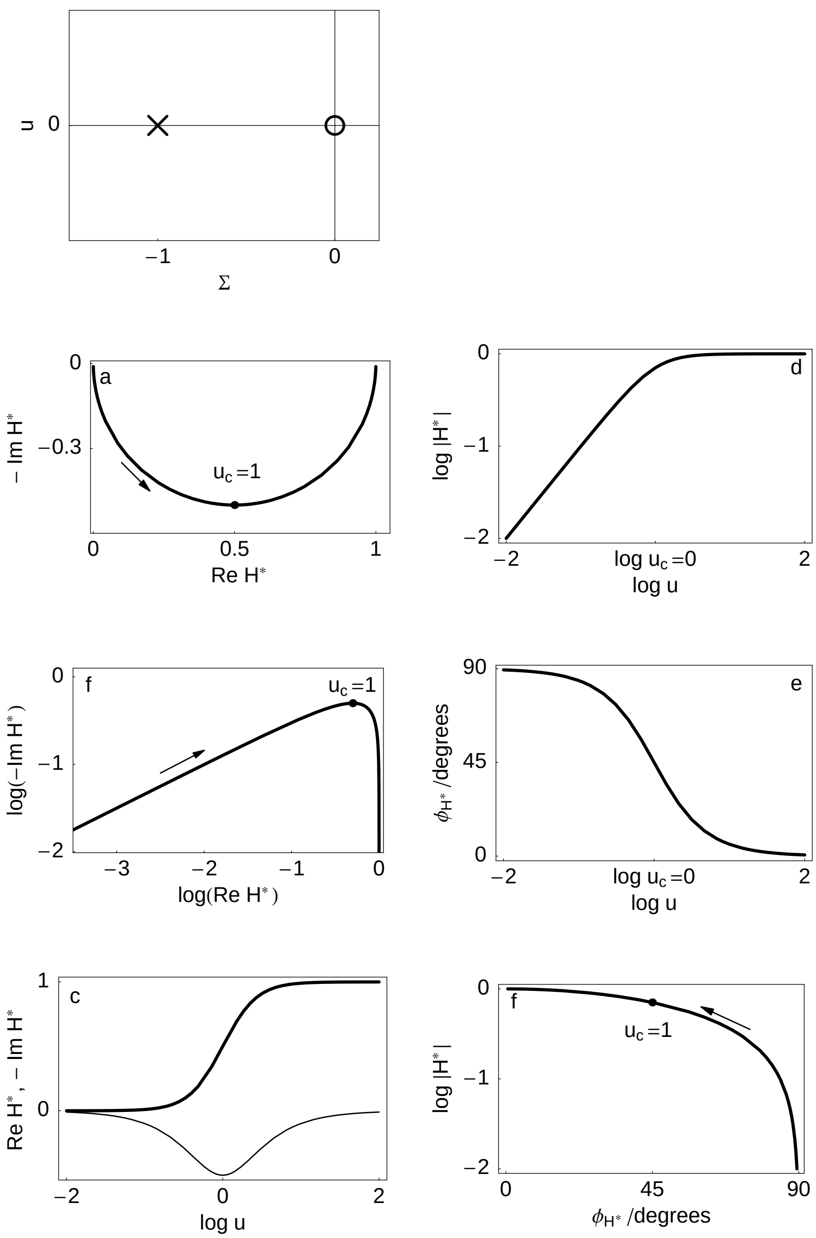

Figure 2.2: Cpz, Nyquist (a), log Nyquist (b) Re H^* *vs.* $\log u$ (c, thick line), $-\text{Im}\ H^*$ *vs.* $\log u$ (c, thin line), Bode (modulus (c) and phase (d)) and Black diagrams of the high-pass first-order transfer function. Arrow always indicates increasing angular frequencies.

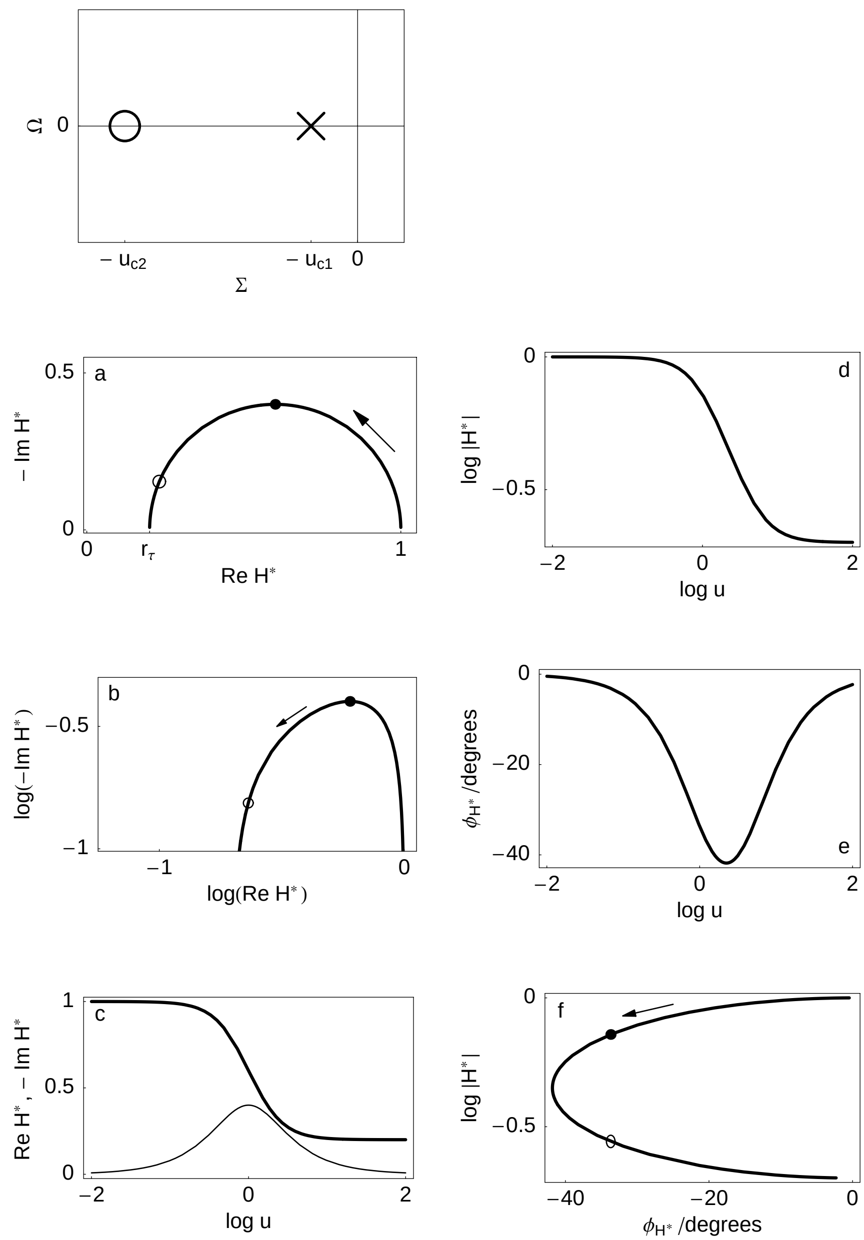

Figure 2.3: Cpz, Nyquist (a), log Nyquist (b) Re H^* *vs.* $\log u$ (c, thick line), $-\mathrm{Im}\ H^*$ *vs.* $\log u$ (c, thin line), Bode (modulus (c) and phase (d)) and Black diagrams of the generalized first order transfer function. $r_\tau = 0.2$ ($r_\tau < 1$), dot: $u_{c1} = 1$, circle: $u_{c2} = 1/r_\tau$.

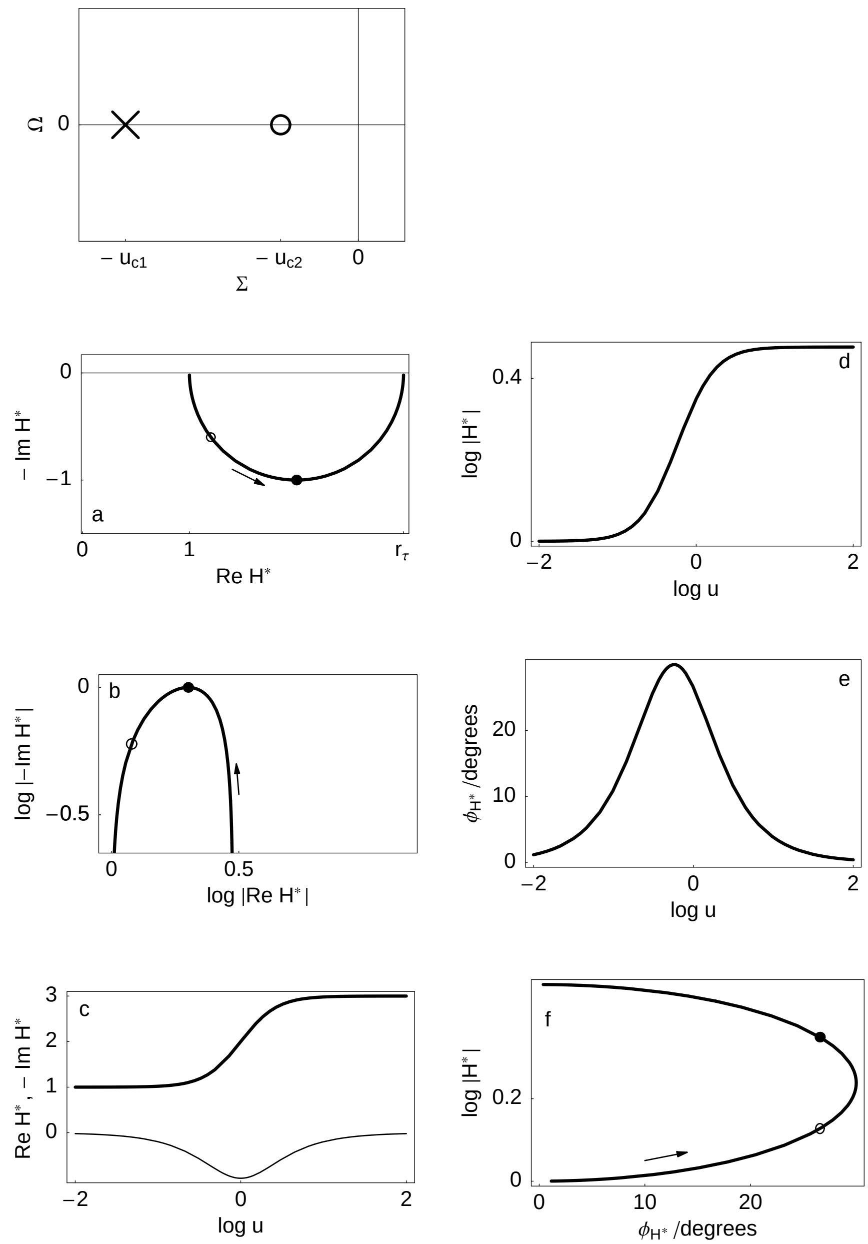

Figure 2.4: Cpz, Nyquist (a), log Nyquist (b) Re H^* *vs.* $\log u$ (c, thick line), $-\mathrm{Im}\ H^*$ *vs.* $\log u$ (c, thin line), Bode (modulus (c) and phase (d)) and Black diagrams of the generalized first order transfer function. $r_\tau = 3$ ($r_\tau > 1$), dot: $u_{c1} = 1$, circle: $u_{c2} = 1/r_\tau$.

Chapter 3

Second-order and generalized second-order transfer functions

3.1 Second-order transfer function

$$H(s) = \frac{K}{1 + a_1\, s + a_2\, s^2}$$

3.1.1 Second-order with real poles

$$H(s) = \frac{K}{(1 + \tau_1\, s)\,(1 + \tau_2\, s)},\ H(\omega) = \frac{K}{(1 + \tau_1\,\mathrm{i}\,\omega)\,(1 + \tau_2\,\mathrm{i}\,\omega)}$$

$$H^*(S) = \frac{H(s)}{K} = \frac{1}{(1 + S)\,(1 + r_\tau\, S)},\ S = \tau_1\, s = \Sigma + \mathrm{i}\,u,\ \Sigma = \tau_1\,\sigma,\ u = \tau_1\, w,\ \ (\tau_1 > \tau_2)$$

Two real poles: $S_{\mathrm{p1}} = 1 = -u_{\mathrm{c1}}$, $S_{\mathrm{p1}} = 1/r_\tau = -u_{\mathrm{c1}}$

$$H^*(u) = \frac{1}{(1 + \mathrm{i}\,u)\,(1 + r_\tau\,\mathrm{i}\,u)}$$

$$\mathrm{Re}\ H^*(u) = \frac{1 - u^2\, r_\tau}{(1 + u^2)\ (1 + u^2\, {r_\tau}^2)},\ \mathrm{Im}\ H^*(u) = -\frac{u\ (1 + r_\tau)}{(1 + u^2)\ (1 + u^2\, {r_\tau}^2)}$$

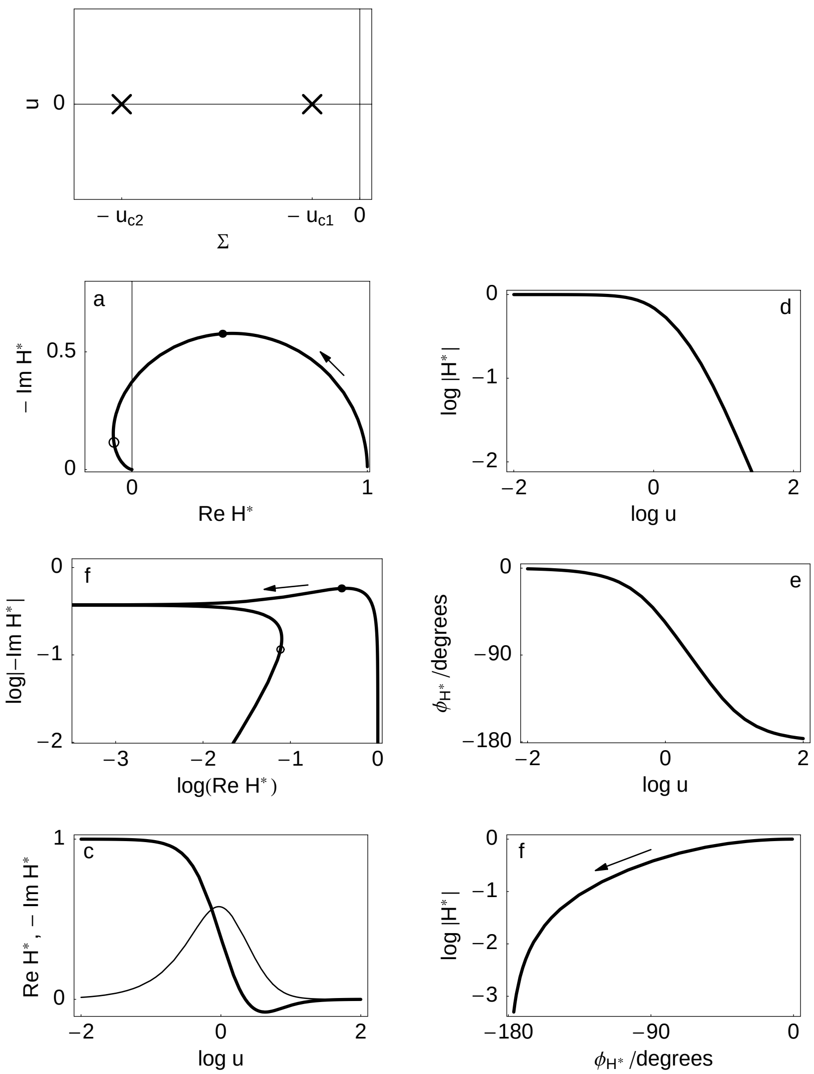

Figure 3.1: Cpz, Nyquist (a), log Nyquist (b) Re H^* *vs.* $\log u$ (c, thick line), $-$Im H^* *vs.* $\log u$ (c, thin line), Bode (modulus (c) and phase (d)) and Black diagrams of the second order transfer function with real poles. $r_\tau = 0.2$ ($r_\tau < 1$), dot: $u_{c1} = 1$, circle: $u_{c2} = 1/r_\tau$.

Part II

Circuit made of impedances

Chapter 4

Circuits made of two impedances

4.1 Circuit ($\mathbf{Z}_1$+$\mathbf{Z}_2$)

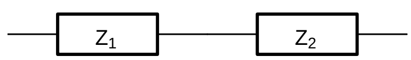

Figure 4.1: Circuit (Z_1+Z_2).

$$Z = Z_1 + Z_2,\ \mathrm{Re}\, Z = \mathrm{Re}\, Z_1 + \mathrm{Re}\, Z_2,\ \mathrm{Im}\, Z = \mathrm{Im}\, Z_1 + \mathrm{Im}\, Z_2$$

4.2 Circuit ($\mathbf{Z}_1$/$\mathbf{Z}_2$)

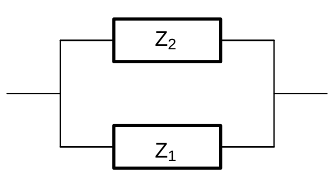

Figure 4.2: Circuit (Z_1/Z_2).

$$Z = \frac{Z_1\, Z_2}{Z_1 + Z_2} \tag{4.1}$$

$$\mathrm{Re}\ Z = \frac{\mathrm{Im}\ {Z_2}^2\, \mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2\ \left(\mathrm{Im}\ {Z_1}^2 + \mathrm{Re}\ Z_1\ (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)\right)}{(ImZ_1 + \mathrm{Im}\ Z_2)^2 + (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)^2} \tag{4.2}$$

$$\mathrm{Im}\ Z = \frac{\mathrm{Im}\ Z_2\ \left(\mathrm{Im}\ Z_1\ (\mathrm{Im}\ Z_1 + \mathrm{Im}\ Z_2) + \mathrm{Re}\ {Z_1}^2\right) + \mathrm{Im}\ Z_1\, \mathrm{Re}\ {Z_2}^2}{(\mathrm{Im}\ Z_1 + \mathrm{Im}\ Z_2)^2 + (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)^2} \tag{4.3}$$

Chapter 5

Circuits made of three impedances

5.1 Circuit $((\mathbf{Z}_1/\mathbf{Z}_2)+\mathbf{Z}_3)$

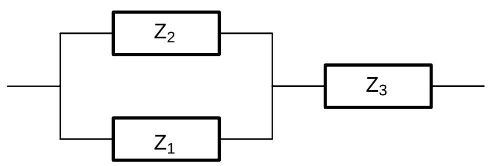

Figure 5.1: Circuit $((Z_1/Z_2)+Z_3)$.

$$Z = \frac{Z_1\,Z_2 + Z_1\,Z_3 + Z_2\,Z_3}{Z_1 + Z_2} \tag{5.1}$$

$$\mathrm{Re}\ Z = \frac{\mathrm{Im}\ {Z_2}^2\,\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2\ \left(\mathrm{Im}\ {Z_1}^2 + \mathrm{Re}\ Z_1\ (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)\right)}{(\mathrm{Im}\ Z_1 + \mathrm{Im}\ Z_2)^2 + (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)^2} + \mathrm{Re}\ Z_3 \tag{5.2}$$

$$\mathrm{Im}\ Z = \frac{\mathrm{Im}\ Z_2\ \left(\mathrm{Im}\ Z_1\ (\mathrm{Im}\ Z_1 + \mathrm{Im}\ Z_2) + \mathrm{Re}\ {Z_1}^2\right) + \mathrm{Im}\ Z_1\,\mathrm{Re}\ {Z_2}^2}{(\mathrm{Im}\ Z_1 + \mathrm{Im}\ Z_2)^2 + (\mathrm{Re}\ Z_1 + \mathrm{Re}\ Z_2)^2} + \mathrm{Im}\ Z_3 \tag{5.3}$$

5.2 Circuit $((\mathbf{Z}_1+\mathbf{Z}_2)/\mathbf{Z}_3)$

$$Z = \frac{(Z_1 + Z_2)\ Z_3}{Z_1 + Z_2 + Z_3}$$

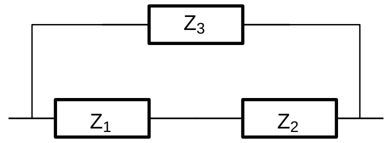

Figure 5.2: Circuit $((Z_1+Z_2)/Z_3)$.

$$\begin{aligned}\text{Re } Z = \Big(& (\text{Im } Z_1 + \text{Im } Z_2)^2 \text{ Re } Z_3 + \\ & (\text{Re } Z_1 + \text{Re } Z_2) \left(\text{Im } Z_3{}^2 + \text{Re } Z_3 \left(\text{Re } Z_1 + \text{Re } Z_2 + \text{Re } Z_3\right)\right)\Big) / \\ & \Big((\text{Im } Z_1 + \text{Im } Z_2 + \text{Im } Z_3)^2 + (\text{Re } Z_1 + \text{Re } Z_2 + \text{Re } Z_3)^2\Big)\end{aligned}$$

$$\begin{aligned}\text{Im } Z = \Big(& \text{Im } Z_3 \Big((\text{Im } Z_1 + \text{Im } Z_2) (\text{Im } Z_1 + \text{Im } Z_2 + \text{Im } Z_3) + (\text{Re } Z_1 + \text{Re } Z_2)^2\Big) + \\ & (\text{Im } Z_1 + \text{Im } Z_2) \text{ Re } Z_3{}^2\Big) / \\ & \Big((\text{Im } Z_1 + \text{Im } Z_2 + \text{Im } Z_3)^2 + (\text{Re } Z_1 + \text{Re } Z_2 + \text{Re } Z_3)^2\Big)\end{aligned}$$

Chapter 6

Circuits made of four impedances

6.1 Circuit $((\mathbf{Z}_1/\mathbf{Z}_2)+(\mathbf{Z}_3/\mathbf{Z}_4))$

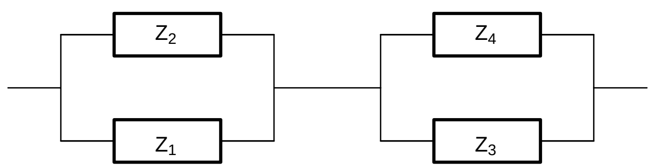

Figure 6.1: Circuit $((Z_1/Z_2)+(Z_3/Z_4))$.

$$Z = \frac{Z_1\,Z_2\,Z_3 + Z_1\,Z_2\,Z_4 + Z_1\,Z_3\,Z_4 + Z_2\,Z_3\,Z_4}{(Z_1+Z_2)\,(Z_3+Z_4)}$$

6.2 Circuit $((\mathbf{Z}_1+(\mathbf{Z}_2/\mathbf{Z}_3))/\mathbf{Z}_4)$

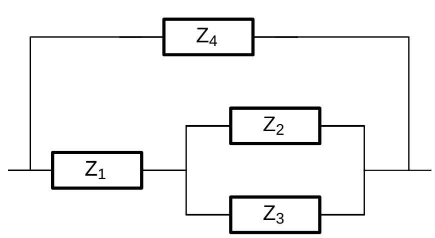

Figure 6.2: Circuit $((Z_1+(Z_2/Z_3))/Z_4)$.

$$Z = \frac{(Z_1\,Z_2 + Z_1\,Z_3 + Z_2\,Z_3)\;Z_4}{Z_1\,Z_2 + Z_1\,Z_3 + Z_2\,Z_3 + Z_2\,Z_4 + Z_3\,Z_4}$$

6.3 Circuit $(((\mathbf{Z}_1+\mathbf{Z}_2)/\mathbf{Z}_3)/\mathbf{Z}_4)$

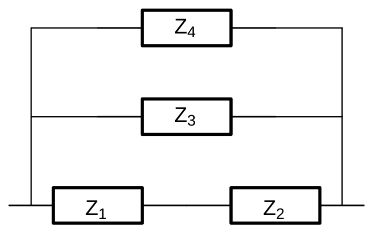

Figure 6.3: Circuit $(((Z_1+Z_2)/Z_3)/Z_4)$.

$$Z = \frac{(Z_1 + Z_2)\;Z_3\,Z_4}{Z_1\,Z_3 + Z_2\,Z_3 + Z_1\,Z_4 + Z_2\,Z_4 + Z_3\,Z_4}$$

Bibliography

[1] Fournier, J., Wrona, P. K., Lasia, A., Lacasse, R., Lalancette, J.-M., Menard, H., and Brossard, L. *J. Electrochem. Soc. 139* (1992), 2372.

[2] Gille, J.-C., Decaulne, P., and Pélegrin, M. *Dynamique de la commande linéaire*. Dunod, Paris, 1991.